

A new family realizing saturated fusion systems

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Abstract

We construct a new group model for fusion systems related to Robinson's models and study modules and a homology decomposition associated with it. Moreover we prove analogues of Glaubermann's and Thompson's theorems for p -local finite groups and a Kuenneth formula for fusion systems.

1 Introduction

In the topological theory of p -local finite groups introduced by Broto, Levi and Oliver one tries to approximate the classifying space of a finite group via the p -local structure of the group, at least up to \mathbb{F}_p -cohomology. In this article we introduce a new group model related to Robinson's construction and study a homology decomposition and a family of modules associated to it. Building on the work of Diaz, Glesser, Mazza and Park we prove analogues of Glaubermann's and Thompson's theorems for p -local finite groups. Moreover we provide a Kuenneth formula independently of the existence of a classifying space.

2 Preliminaries

2.1 Fusion Systems

We review the basic definitions of fusion systems and centric linking systems and establish our notations. Our main references are [4], [5] and [12]. Let S be a finite p -group. A fusion system \mathcal{F} on S is a category whose objects are all the subgroups of S , and which satisfies the following two properties for all $P, Q \leq S$: The set $\text{Hom}_{\mathcal{F}}(P, Q)$ contains injective group homomorphisms and amongst them all morphisms induced by conjugation of elements in S and each element is the composite of an isomorphism in \mathcal{F} followed by an inclusion. Two subgroups $P, Q \leq S$ will be called \mathcal{F} -conjugate if they are isomorphic in \mathcal{F} . Define $\text{Out}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{F}}(P)/\text{Inn}(P)$ for all $P \leq S$. A subgroup $P \leq S$ is fully centralized resp. fully normalized in \mathcal{F} if $|C_S(P)| \geq |C_S(P')|$ resp. $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is \mathcal{F} -conjugate to P . \mathcal{F} is called saturated if for all $P \leq S$ which is fully normalized in \mathcal{F} , P is fully centralized in \mathcal{F} and $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(P))$ and moreover if $P \leq S$ and $\phi \in \text{Hom}_{\mathcal{F}}(P, S)$ are such that $\phi(P)$ is fully centralized, and if we set $N_{\phi} = \{g \in N_S(P) \mid \phi c_g \phi^{-1} \in \text{Aut}_S(\phi(P))\}$, then there is $\bar{\phi} \in \text{Hom}_{\mathcal{F}}(N_{\phi}, S)$ such that $\bar{\phi}|_P = \phi$. A subgroup $P \leq S$ will be called \mathcal{F} -centric if $C_S(P') \leq P'$ for all P' which are \mathcal{F} -conjugate to P . Denote \mathcal{F}^c the full subcategory of \mathcal{F} with objects the \mathcal{F} -centric subgroups of S . Let $\mathcal{O}(\mathcal{F})$ be the orbit category of \mathcal{F} with objects the same objects as \mathcal{F} and morphisms the set $\text{Mor}_{\mathcal{O}(\mathcal{F})}(P, Q) = \text{Mor}_{\mathcal{F}}(P, Q)/\text{Inn}(Q)$. Let $\mathcal{O}^c(\mathcal{F})$ be the full subcategory of $\mathcal{O}(\mathcal{F})$ with objects the \mathcal{F} -centric subgroups of \mathcal{F} . A centric linking system associated to \mathcal{F} is a category \mathcal{L} whose objects are the \mathcal{F} -centric subgroups of S , together with a functor $\pi : \mathcal{L} \rightarrow \mathcal{F}^c$, and "distinguished"

monomorphisms $\delta_P : P \rightarrow \text{Aut}_{\mathcal{L}}(P)$ for each \mathcal{F} -centric subgroup $P \leq S$ such that the following conditions are satisfied: π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\text{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \text{Aut}_{\mathcal{L}}(P)$), and π induces a bijection $\text{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\sim} \text{Hom}_{\mathcal{F}}(P, Q)$. For each \mathcal{F} -centric subgroup $P \leq S$ and each $x \in P$, $\pi(\delta_P(x)) = c_x \in \text{Aut}_{\mathcal{F}}(P)$. For each $f \in \text{Mor}_{\mathcal{L}}(P, Q)$ and each $x \in P$, $f \circ \delta_P(x) = \delta_Q(\pi f(x)) \circ f$. Let $\mathcal{F}, \mathcal{F}'$ be fusion systems on finite p -groups S, S' , respectively. A morphism of fusion systems from \mathcal{F} to \mathcal{F}' is a pair (α, Φ) consisting of a group homomorphism $\alpha : S \rightarrow S'$ and a covariant functor $\Phi : \mathcal{F} \rightarrow \mathcal{F}'$ with the following properties: for any subgroup Q of S we have $\alpha(Q) = \Phi(Q)$ and for any morphism $\phi : Q \rightarrow R$ in \mathcal{F} we have $\Phi(\phi) \circ \alpha|_Q = \alpha|_R \circ \phi$. Let G be a discrete group. A finite subgroup S of G will be called a Sylow p -subgroup of G if S is a p -subgroup of G and all p -subgroups of G are conjugate to a subgroup of S . A group G is called p -perfect if $H_1(BG; \mathbb{Z}_p) = 0$. Let \mathcal{F} be a saturated fusion system over the finite p -group S . Let G_1, G_2 be groups with Sylow p -subgroups and $\phi : G_1 \rightarrow G_2$ a group homomorphism. ϕ will be called fusion preserving if the restriction to the respective Sylow p -subgroups induces an isomorphism of fusion systems $\mathcal{F}_{S_1}(G_1) \cong \mathcal{F}_{S_2}(G_2)$. Let S be a finite p -group and let $P_1, \dots, P_r, Q_1, \dots, Q_r$ be subgroups of S . Let ϕ_1, \dots, ϕ_r be injective group homomorphisms $\phi_i : P_i \rightarrow Q_i \ \forall i$. The fusion system generated by ϕ_1, \dots, ϕ_r is the minimal fusion system \mathcal{F} over S containing ϕ_1, \dots, ϕ_r . Let \mathcal{F} be a fusion system on a finite p -group S . A subgroup $T \leq S$ is strongly closed in S with respect to \mathcal{F} , if for each subgroup P of T , each $Q \leq S$, and each $\phi \in \text{Mor}_{\mathcal{F}}(P, Q)$, $\phi(P) \leq T$. Fix any pair $S \leq G$, where G is a (possibly infinite) group and S is a finite p -subgroup. Define $\mathcal{F}_S(G)$ to be the category whose objects are the subgroups of S , and where $\text{Mor}_{\mathcal{F}_S(G)}(P, Q) = \text{Hom}_G(P, Q) = \{c_g \in \text{Hom}(P, Q) | g \in G, gPg^{-1} \leq Q\} \cong N_G(P, Q)/C_G(P)$. Here c_g denotes the homomorphism conjugation by g ($x \mapsto gxg^{-1}$), and $N_G(P, Q) = \{g \in G | gPg^{-1} \leq Q\}$ (the transporter set). For each $P \leq S$, let $C'_G(P)$ be the maximal p -perfect subgroup of $C_G(P)$. Let $\mathcal{L}_S^c(G)$ be the category whose objects are the $\mathcal{F}_S(G)$ -centric subgroups of S , and where $\text{Mor}_{\mathcal{L}_S^c(G)}(P, Q) = N_G(P, Q)/C'_G(P)$. Let $\pi : \mathcal{L}_S^c(G) \rightarrow \mathcal{F}_S(G)$ be the functor which is the inclusion on objects and sends the class of $g \in N_G(P, Q)$ to conjugation by g . For each $\mathcal{F}_S(G)$ -centric subgroup $P \leq G$, let $\delta_P : P \rightarrow \text{Aut}_{\mathcal{L}_S^c(G)}(P)$ be the monomorphism induced by the inclusion $P \leq N_G(P)$. A triple $(S, \mathcal{F}, \mathcal{L})$ where S is a finite p -group, \mathcal{F} is a saturated fusion system on S , and \mathcal{L} is an associated centric linking system to \mathcal{F} , is called a p -local finite group. It's classifying space is $|\mathcal{L}|_p^\wedge$ where $(-)_p^\wedge$ denotes the p -completion functor in the sense of Bousfield and Kan. This is partly motivated by the fact that every finite group G gives canonically rise to a p -local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ and $BG_p^\wedge \simeq |\mathcal{L}_p^\wedge[1]$. In particular, all fusion systems coming from finite groups are saturated. Let \mathcal{F} be a fusion system on the finite p -group S . \mathcal{F} is called an Alperin fusion system if there are subgroups P_1, P_2, \dots, P_r of S and finite groups L_1, \dots, L_r such that: $P_i \cong O_p(L_i)$ (the largest normal p -subgroup of L_i) and $C_{L_i}(P_i) = Z(P_i)$, $L_i/P_i \cong \text{Out}_{\mathcal{F}}(P_i)$ for each i , $N_S(P_i)$ is a Sylow p -subgroup of L_i for each i and $P_1 = S$, for each i $\mathcal{F}_{N_S(P_i)}(L_i)$ is contained in \mathcal{F} , \mathcal{F} is generated by all the $\mathcal{F}_{N_S(P_i)}(L_i)$. Recall that every saturated fusion system is Alperin since let \mathcal{F} be a saturated fusion system over a finite p -group S . Let $S = P_1, \dots, P_n$ be subgroups of S which are representatives of isomorphism classes of centric radicals in \mathcal{F} . From Section 4 in [2] it follows that we can find corresponding groups L_i , $i = 1, \dots, n$ which have all the properties.

One can define fusion systems and centric linking systems in a topological setting. We will need this when we make use of the fact that a group realizes a given fusion system if and only if its classifying space has a certain homotopy type. In particular we have for a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ and a group G such that $\mathcal{F}_S(G) = \mathcal{F}$ that there is a map from the one-skeleton of the nerve of \mathcal{L} to the classifying space : $|\mathcal{L}|^{(1)} \rightarrow BG$. Fix a space X , a finite p -group S , and a map $f : BS \rightarrow X$. Define $\mathcal{F}_{S,f}(X)$ to be the category whose objects are the subgroups of S , and whose morphisms are given by $\text{Hom}_{\mathcal{F}_{S,f}(X)}(P, Q) = \{\phi \in \text{Inj}(P, Q) | f|_{BP} \simeq f|_{BQ} \circ B\phi\}$ for each $P, Q \leq S$. Define $\mathcal{F}'_{S,f}(X) \subseteq \mathcal{F}_{S,f}(X)$ to be the subcategory with the same objects as $\mathcal{F}_{S,f}(X)$, and where $\text{Mor}_{\mathcal{F}'_{S,f}(X)}(P, Q)$ (for $P, Q \leq S$) is the set of all composites of restrictions of morphisms in $\mathcal{F}_{S,f}(X)$ between $\mathcal{F}_{S,f}(X)$ -centric subgroups. Define $\mathcal{L}_{S,f}^c(X)$ to be the category whose objects are the $\mathcal{F}_{S,f}(X)$ -centric subgroups of S , and whose morphisms are defined by $\text{Mor}_{\mathcal{L}_{S,f}^c(X)}(P, Q) =$

$\{(\phi, [H]) | \phi \in \text{Inj}(P, Q), H : BP \times I \rightarrow X, H|_{BP \times 0} = f|_{BP}, H|_{BP \times 1} = f|_{BQ} \circ B\phi\}$. The composite in $\mathcal{L}_{S,f}^c(X)$ of morphisms $P \xrightarrow{(\phi, [H])} Q \xrightarrow{(\psi, [K])} R$, where $H : BP \times I \rightarrow X$ and $K : BQ \times I \rightarrow X$ are homotopies as described above, are defined by setting $(\psi, [K]) \circ (\phi, [H]) = (\psi \circ \phi, [(K \circ (B\phi \times ID)) * H])$, where $*$ denotes composition (juxtaposition) of homotopies. Let $\pi : \mathcal{L}_{S,f}^c(X) \rightarrow \mathcal{F}_{S,f}(X)$ be the forgetful functor: it is the inclusion on objects, and sends a morphism $(\phi, [H])$ to ϕ . For each $\mathcal{F}_{S,f}(X)$ -centric subgroup $P \leq S$, let $\delta_P : P \rightarrow \text{Aut}_{\mathcal{L}_{S,f}^c(X)}(P)$ be the "distinguished homomorphism" which sends $g \in P$ to $(c_g, [f|_{BP} \circ H_g])$, where $H_g : BP \times I \rightarrow BP$ denotes the homotopy from Id_{BP} to Bc_g induced by the natural transformation of functors $\mathcal{B}(G) \rightarrow \mathcal{B}(G)$ which sends the unique object \circ_G in $\mathcal{B}G$ to the morphism \hat{g} corresponding to g in G .

Theorem 2.1 ([5], Theorem 2.1.) *Fix a space X , a p -group S , and a map $f : BS \rightarrow X$. Assume that f is Sylow; $f|_{BP}$ is a centric map for each $\mathcal{F}_{S,f}(X)$ -centric subgroup $P \leq S$; and every $\mathcal{F}'_{S,f}(X)$ -centric subgroup of S is also $\mathcal{F}_{S,f}(X)$ -centric. Then the triple $(S, \mathcal{F}'_{S,f}(X), \mathcal{L}_{S,f}^c(X))$ is a p -local finite group.*

2.2 Groups Realizing a Given Fusion System

Given a fusion system \mathcal{F} on a finite p -group S it is not always true that there exists a finite group G such that $\mathcal{F}_S = \mathcal{F}_S(G)$, (see [4], chapter 9 for example). However for every fusion system \mathcal{F} there exists an infinite group \mathcal{G} such that $\mathcal{F}_S(\mathcal{G}) = \mathcal{F}$. We now describe the constructions by G. Robinson [14], and I. Leary and R. Stancu [8]. The groups of Robinson type are iterated amalgams of automorphism groups in the linking system, if it exists, over the S -normalizers of the respective \mathcal{F} -centric subgroups of S . Note that these automorphism groups exist and are known regardless of whether \mathcal{L} exists or not.

Theorem 2.2 ([14], Theorem 2.) *Let \mathcal{F} be an Alperin fusion system on a finite p -group S and associated groups L_1, \dots, L_n as in the definition. Then there is a finitely generated group \mathcal{G} which has S as a Sylow p -subgroup such that the fusion system \mathcal{F} is realized by \mathcal{G} . The group \mathcal{G} is given explicitly by $\mathcal{G} = L_1 \underset{N_S(P_2)}{*} L_2 \underset{N_S(P_3)}{*} \dots \underset{N_S(P_n)}{*} L_n$ with L_i, P_i as in the definition.*

Corresponding to the various versions of Alperin's fusion theorem (essential subgroups, centric subgroups, centric radical subgroups) there exist several canonical choices for the groups generating \mathcal{F} . The group constructed by I. Leary and R. Stancu is an iterated HNN-construction.

Theorem 2.3 ([8], Theorem 2.) *Suppose that \mathcal{F} is the fusion system on S generated by $\Phi = \{\phi_1, \dots, \phi_r\}$. Let T be a free group with free generators t_1, \dots, t_r , and define G as the quotient of the free product $S * T$ by the relations $t_i^{-1} u t_i = \phi_i(u)$ for all i and for all $u \in P_i$. Then S embeds as a p -Sylow subgroup of G and $\mathcal{F}_S(G) = \mathcal{F}$.*

2.3 Graphs of Groups

We give a short introduction to graphs of groups stating results we need. A finite directed graph Γ consists of two sets, the vertices V and the directed edges E , together with two functions $\iota, \tau : E \rightarrow V$. For $e \in E$, $\iota(e)$ is called the initial vertex of e and $\tau(e)$ is the terminal vertex of e . Multiple edges and loops are allowed in this definition. The graph Γ is connected if the only equivalence relation on V that contains all $(\iota(e), \tau(e))$ is the relation with just one class. A graph Γ may be viewed as a category, with objects the disjoint union of V and E and two non-identity morphisms with domain e for each $e \in E$, one morphism $e \rightarrow \iota(e)$ and one morphism $e \rightarrow \tau(e)$. A graph Γ of groups is a connected graph Γ together with groups G_v, G_e for each vertex and edge and injective group homomorphisms $f_{e,\iota} : G_e \rightarrow G_{\iota(e)}$ and $f_{e,\tau} : G_e \rightarrow G_{\tau(e)}$ for each edge e .

3 A new family realizing saturated fusion systems

We introduce a new group model realizing saturated fusion systems related to the construction of G. Robinson.

Theorem 3.1 *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and*

$$\mathcal{G} = L_1 \underset{N_S(P_2)}{*} L_2 \underset{N_S(P_3)}{*} \dots \underset{N_S(P_n)}{*} L_n$$

a model of Robinson type for \mathcal{F} . For each of the L_i , $i = 1, \dots, n$ choose subgroups K_1, \dots, K_m of L_i such that each K_j contains (an isomorphic copy of) the group $N_S(P_i)$ and K_1, \dots, K_m generate the group L_i . Assume as we can that $S \in \text{Syl}_p(K_1)$ and after reindexing let G be the iterated amalgam

$$G = K_1 \underset{N_S(P_2)}{*} K_2 \underset{N_S(P_3)}{*} \dots \underset{N_S(P_n)}{*} K_n.$$

Then G contains S as a Sylow p -subgroup and $\mathcal{F}_S(G) = \mathcal{F}$.

Theorem 3.2 *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and let $\Phi = \{\phi_1, \dots, \phi_n\}$ be a subset of the set of automorphisms of the fusion system which is chosen in a minimal way, i. e. we cannot omit any element without obtaining a proper subsystem. Moreover assume as we can that all the elements of Φ have order coprime to p . Then the group*

$$\mathcal{G} := S * F(\Phi) / \langle \phi u \phi^{-1} = \phi(u), \phi^{\deg(\phi)} = 1 \rangle$$

has the following properties. The group $S \in \text{Syl}_p(\mathcal{G})$, $\mathcal{F}_S(\mathcal{G}) = \mathcal{F}$, the classifying space $B\mathcal{G}$ is p -good and the cohomology of $B\mathcal{G}$ is F -isomorphic in the sense of Quillen to the stable elements.

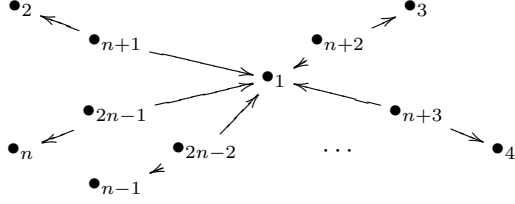
3.1 Homology Decompositions

We investigate the cohomology of our models for a saturated fusion system \mathcal{F} over a finite p -group S . In the following \mathcal{G} will always be a model for \mathcal{F} of Robinson type, i. e. $\mathcal{G} = L_1 \underset{N_S(P_2)}{*} L_2 \underset{N_S(P_3)}{*} \dots \underset{N_S(P_n)}{*} L_n$ where L_1, \dots, L_n , P_1, \dots, P_n can be chosen such that P_1, \dots, P_n are representatives of isomorphism classes of centric radicals, of \mathcal{F} -centrics or of essential subgroups of \mathcal{F} , and L_1, \dots, L_n are the corresponding automorphism groups of P_1, \dots, P_n in the linking system if it exists. Note that these groups are known and are unique and do exist regardless of whether \mathcal{L} exists or not, see [12, Theorem 4.6.] following the discussion in [2], Section 4.

Theorem 3.3 *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and \mathcal{G} a discrete group such that $S \in \text{Syl}_p(\mathcal{G})$ and $\mathcal{F} = \mathcal{F}_S(\mathcal{G})$. Then there exist a natural map of unstable algebras $H^*(B\mathcal{G}) \xrightarrow{q} H^*(\mathcal{F})$ making $H^*(\mathcal{F})$ a module over $H^*(B\mathcal{G})$.*

Theorem 3.4 *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and \mathcal{G} a model of our type for \mathcal{F} . Then there exist natural maps of unstable algebras over the Steenrod algebra $H^*(B\mathcal{G}) \xrightarrow{q} H^*(|\mathcal{L}|)$ and $H^*(|\mathcal{L}|) \xrightarrow{r} H^*(B\mathcal{G})$ such that we obtain a split short exact sequence of unstable modules over the Steenrod algebra $0 \rightarrow H^*(|\mathcal{L}|) \xrightarrow[r]{q} H^*(B\mathcal{G}) \xrightarrow{\pi} W \rightarrow 0$ where $W \cong \text{Ker}(\text{Res}_S^G)$.*

Proof: Let \mathcal{C} be the following category.



Denote by $\phi_{i,j} : \bullet_i \rightarrow \bullet_j$ the unique morphism

in \mathcal{C} between \bullet_i and \bullet_j if it exists. Let $F : \mathcal{C} \rightarrow \text{Spaces}$ be a functor with $F(\bullet_i) = BL_i$ for $i = 1, \dots, n$, $BN_S(P_i)$ for $i = n+1, \dots, 2n-1$ and $F(\phi_{i,j}) = Bincl : F(\bullet_i) \rightarrow F(\bullet_j)$ for all $\phi_{i,j} : \bullet_i \rightarrow \bullet_j$ in \mathcal{C} , $i = n+1, \dots, 2n-1$, $j = i-n+1, 1$. Note that $\text{hocolim}_{\mathcal{C}}(F)$ is a $K(G, 1)$. Since $K_i \leq L_i = \text{Aut}_{\mathcal{L}}(P_i)$ for all $i = 1, \dots, n$ we have a functor $\mathcal{BK}_i \rightarrow \mathcal{L}$ which sends the unique object \bullet to P_i and a morphism x to the corresponding morphism in $\text{Aut}_{\mathcal{L}}(P_i)$ for all $i = 1, \dots, n$. Therefore we obtain a map BK_i to $|\mathcal{L}|$ for all $i = 1, \dots, n$. Note that all the diagrams

$$\begin{array}{ccc} & Bincl & BK_1 \\ BN_S(P_i) & \searrow & \searrow \\ & Bincl & BK_i \end{array} \quad \begin{array}{c} \text{commute up} \\ | \mathcal{L} \end{array}$$

to homotopy since the third axiom from the definition of the linking system guarantees that we can find a compatible system of lifts of the inclusion $\iota_{N_S(P_i), S}$ in \mathcal{L} for all $i = 1, \dots, n$ such that all the diagrams

$$\begin{array}{ccc} & Bincl & BK_1 \\ BN_S(P_i) & \searrow & \searrow \\ & Bincl & BK_i \end{array} \quad \mathcal{L}$$

$\bullet \in \text{Obj}(BN_S(P_i))$ to $\iota_{N_S(P_i), S}$ for $i = 1, \dots, n$. We obtain a map from the 1-skeleton of the homotopy colimit of the functor F over the category \mathcal{C} to $|\mathcal{L}|$. Since \mathcal{C} is a 1-dimensional category we obtain a map from BG to $|\mathcal{L}|$. This map will be denoted by q inducing $H^*(|\mathcal{L}|) \xrightarrow{q^*} H^*(BG)$. Denote the kernel of the map f by W . We have the following commutative diagram of unstable algebras over the Steenrod algebra where the maps q^* and $incl$ are injective

$$\begin{array}{ccc} H^*(|\mathcal{L}|) & \xrightarrow{q^*} & H^*(BG) \\ \text{incl} \searrow & & \downarrow \text{Res}_S^G \\ & & H^*(BS). \end{array} \quad \text{Commutativity}$$

implies that $W \cong \text{Ker}(\text{Res}_S^G)$ in the category of unstable modules over the Steenrod algebra. \square

Theorem 3.5 *Let \mathcal{F} be an Alperin fusion system and G a model of our type for it. Then BG is p -good.*

Proof: The group G is a finite amalgam of finite groups. Note that each K_i is generated by $N_S(P_i)$ and elements of p' -order. Therefore G is generated by elements of p' -order and S . Let K be the subgroup of G generated by all elements of p' -order. Note that K is normal in G and S surjects on G/K and therefore G/K is a finite p -group. We have $H^1(BK; \mathbb{F}_p) = 0$ and therefore K is p -perfect. Let X be the cover of BG with fundamental group K . Then X is p -good and X_p^\wedge is simply connected since $\pi_1(X)$ is p -perfect as follows from [1, VII.3.2]. Hence $X_p^\wedge \rightarrow BG_p^\wedge \rightarrow B(G/K)$ is a fibration sequence and BG_p^\wedge is p -complete by [1, II.5.2(iv)]. So BG is p -good. \square

Theorem 3.6 *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and G be a model of our type for it. Then $H^*(BG)$ is finitely generated.*

Proof: Note that we have a map $BG = \text{hocolim}_{\mathcal{C}}(F) \rightarrow |\mathcal{L}|$ where F and \mathcal{C} are as defined in the proof of Theorem 4.3. for the model of our type G . Note that $N_S(P_i) \in \text{Syl}_p(K_i)$ for all $i = 1, \dots, n$. It follows from [3, Lemma 2.3.] and [4, Theorem 4.4.(a)] that $H^*(B(P_i))$ is finitely generated

over $H^*(|\mathcal{L}|)$ for all $i = 1, \dots, n$, and $H^*(|\mathcal{L}|)$ is noetherian as follows from [4, Proposition 1.1. and Theorem 5.8.]. Therefore the Bousfield-Kan spectral sequence for $H^*(BG)$ is a spectral sequence of finitely generated $H^*(|\mathcal{L}|)$ -modules, the E_2 term with $E_2^{s,t} = \varinjlim_{\mathcal{C}}^s H^t(F(-); \mathbb{F}_p)$ is concentrated in the first two columns and $E_2 = E_\infty$ for placement reasons. Therefore $H^*(BG)$ is a finitely generated module over $H^*(|\mathcal{L}|)$ and in particular noetherian. \square

3.1.1 A stable retract

Theorem 3.7 *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and G a model of our type for \mathcal{F} . Then $|\mathcal{L}|_p^\wedge$ is a stable retract of BG_p^\wedge .*

Proof. The diagram commutes where q is the map constructed

$$\begin{array}{ccc} \Sigma^\infty B(\delta_S)_p^\wedge & \xrightarrow{\Sigma^\infty BS_p^\wedge} & \Sigma^\infty B\text{incl}_p^\wedge \\ & \searrow & \swarrow \\ \Sigma^\infty |\mathcal{L}|_p^\wedge & \xleftarrow{\Sigma^\infty q_p^\wedge} & \Sigma^\infty BG_p^\wedge \end{array}$$

in the proof of Theorem 4.1. By the work of K. Ragnarsson [13] there is a map $\sigma_{\mathcal{F}} : \Sigma^\infty |\mathcal{L}|_p^\wedge \rightarrow \Sigma^\infty BS_p^\wedge$ such that the composition of maps $\Sigma^\infty |\mathcal{L}|_p^\wedge \xrightarrow{\sigma_{\mathcal{F}}} \Sigma^\infty BS_p^\wedge \xrightarrow{\Sigma^\infty B(\delta_S)_p^\wedge} \Sigma^\infty |\mathcal{L}|_p^\wedge$ is the identity. Since $\Sigma^\infty B(\delta_S)_p^\wedge \circ \sigma_{\mathcal{F}} = \Sigma^\infty q_p^\wedge \circ \Sigma^\infty B\text{incl}_p^\wedge \circ \sigma_{\mathcal{F}}$ we have $|\mathcal{L}|_p^\wedge$ is a stable retract of BG_p^\wedge . \square

4 Modules and Euler characteristic

Let \mathcal{F} be an Alperin fusion system with associated groups L_1, \dots, L_r and associated subgroups K_1, \dots, K_n as described above and let

$$G = K_1 \underset{N_S(P_2)}{*} K_2 \underset{N_S(P_3)}{*} K_3 \cdots \underset{N_S(P_n)}{*} K_n$$

the group model discussed so far. Then we can see inductively that given a group M and group homomorphisms $\phi_i : K_i \rightarrow M$ for $i = 1, \dots, n$ with

$$\text{Res}_{N_S(P_i)}^{K_i}(\phi_i) = \text{Res}_{N_S(P_i)}^S(\phi_1)$$

for each i , there is a unique group homomorphism $\phi : G \rightarrow M$ which extends each ϕ_i .

We now want to study the finite-dimensional kG -modules. Notice that if H is a finite group generated by subgroups M_i for $i = 1, \dots, n$ such that for each i there is a group epimorphism $\alpha_i : K_i \rightarrow M_i$ with $\text{Res}_{N_S(P_i)}^{K_i}(\alpha_i) = \text{Res}_{N_S(P_i)}^S(\alpha_1)$, then H is an epimorphic image of G .

We allow the possibility that $U = O_p(G) \neq 1$. We remark that $U \leq E$, for each i , since

$$N_U(E_i)E_iC_G(E_i)/E_iC_G(E_i) \leq O_p(N_G(E_i))/E_iC_G(E_i) \cong O_p(\text{Out}_{\mathcal{F}}(E_i)) = 1.$$

Theorem 4.1 *For each i , let H_i be a p' -subgroup of K_i , and let $t = \text{lcm}\{[K_i : H_i] : 1 \leq i \leq r\}$. Then there is a group homomorphism $\phi : G \rightarrow S_t$ whose kernel is a free group.*

Proof. It is enough to construct the homomorphism ϕ so that $\ker \phi$ has trivial intersection with S . Let $\Omega = \{1, 2, \dots, t\}$ and let each K_i act as it would on the direct sum of $t/[K_i : H_i]$ copies of the permutation module of K_i on the cosets of H_i . Since S acts semi-regularly, and each $N_S(P_i)$ does, we may label the points so that each $N_S(P_i)$ acts in the manner determined by regarding it as a subgroup of S . By the remarks preceding the Theorem, Ω now has the structure of a G -set. Since the action of P is free, the kernel of the action is a free normal subgroup of finite index. \square

- Theorem 4.2** 1. For $1 \leq i \leq r$ let X_i be a finite-dimensional projective kK_i -module, and suppose that all X_i have equal dimension. Then there is a kG -module X such that $\text{Res}_{K_i}^G(X) \cong X_i$ for each i . Furthermore, $C_G(X)$ is a free normal subgroup of G of finite index.
2. For each i , let V_i be a simple kK_i -module. Then there is a finite dimensional projective kG -module V such that for each i , $\text{soc}(\text{Res}_{K_i}^G(V))$ is isomorphic to a direct sum of copies of V_i .

Proof:

1. Since X_i is a free $kN_S(P_i)$ -module for each i we can suppose that $N_S(P_i)$ has the same action on X_i as it does on $\text{Res}_{N_S(P_i)}^S(X_1)$ for each i . In that case, there is a unique way to extend the action of the K_i on the underlying k -vector space to an action of G on that space. We let X denote the kG -module so obtained. Then X is a free kS -module, so that $C_G(X) \cap S = 1$ and $C_G(X)$ is a free normal subgroup of G of finite index.
2. Let Y_i denote the projective cover of V_i as kK_i -module, so that we also have $\text{soc}(Y_i) \cong V_i$. Let V be a simple kG -module of X . Then $\text{soc}(\text{Res}_{K_i}^G(V))$ is a submodule of $\text{soc}(\text{Res}_{K_i}^G(X))$ for each i , so the result follows. \square

Theorem 4.3 Let \mathcal{F} be a saturated fusion system over the finite p -group S and

$$G = K_1 \underset{N_S(P_2)}{*} K_2 \underset{N_S(P_3)}{*} \cdots \underset{N_S(P_n)}{*} K_n$$

a model of our type for \mathcal{F} . Then we have the following formula

$$\chi(G) = \left(\sum_{1 \leq i \leq n} \frac{1}{|K_i|} \right) - \left(\sum_{2 \leq i \leq n} \frac{1}{|N_S(P_i)|} \right)$$

for the Euler characteristic of $\chi(G)$ of G . Moreover we have

$$\chi(G) = \frac{d_{\mathcal{F}}}{|S| \text{lcm}\{K_i : N_S(P_i), 1 \leq i \leq n\}}$$

for some negative integer $d_{\mathcal{F}}$.

Proof: Proof:

Theorem 4.4 1. Any free subgroup of finite index of X has index divisible by

2.

5 A Kuenneth Formula for Fusion Systems

We prove an analogue of the Kuenneth Formula for saturated fusion systems independently of the existence of a classifying space.

Theorem 5.1 Let $\mathcal{F}_1, \mathcal{F}_2$ be saturated fusion systems over the finite p -groups S_1, S_2 respectively. Then $H^*(\mathcal{F}_1 \times \mathcal{F}_2) \cong H^*(\mathcal{F}_1) \otimes H^*(\mathcal{F}_2)$.

The proof requires a lemma.

Lemma 5.1 *Let $\mathcal{F}_1, \mathcal{F}_2$ be saturated fusion systems over the finite p -groups S_1, S_2 respectively. Let $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ be the saturated fusion system over $S = S_1 \times S_2$. Then $P = P_1 \times P_2$ is \mathcal{F} -centric if and only if P_1 is \mathcal{F}_1 -centric and P_2 is \mathcal{F}_2 -centric.*

Proof: Assume P_1 is \mathcal{F}_1 -centric and P_2 is \mathcal{F}_2 -centric. Then we have for every $P' = P'_1 \times P'_2$ which is \mathcal{F} -conjugate to P that $|C_S(P)| = |C_{S_1 \times S_2}(P_1 \times P_2)| = |C_{S_1}(P_1)| \cdot |C_{S_2}(P_2)| \geq |C_{S_1}(P'_1)| \cdot |C_{S_2}(P'_2)| = |C_S(P')|$. From the precedent inequality it can be seen that the converse holds as well. \square

Proof of the Theorem: The following diagram commutes where the vertical isomorphisms are given through the Kuenneth Formula for topological spaces $H^*(P_1 \times P_2; \mathbb{F}_p) \cong H^*(P_1; \mathbb{F}_p) \otimes H^*(P_2; \mathbb{F}_p)$ and $H^*(Q_1 \times Q_2; \mathbb{F}_p) \cong H^*(Q_1; \mathbb{F}_p) \otimes H^*(Q_2; \mathbb{F}_p)$.

$$\begin{array}{ccc}
& H^*(\mathcal{F}_1 \times \mathcal{F}_2) & \\
& \parallel & \\
& \varinjlim_{\mathcal{O}^c(\mathcal{F}_1 \times \mathcal{F}_2)} H^*(-) & \\
\swarrow & & \searrow \\
H^*(P_1 \times P_2) & \xrightarrow{(\phi_1 \times \phi_2)^*} & H^*(Q_1 \times Q_2) \\
\cong & & \cong \\
H^*(P_1) \otimes H^*(P_2) & \xrightarrow{\phi_1^* \otimes \phi_2^*} & H^*(Q_1) \otimes H^*(Q_2) \\
\swarrow & & \searrow \\
& (\varinjlim_{\mathcal{O}^c(\mathcal{F}_1)} H^*(-)) \otimes (\varinjlim_{\mathcal{O}^c(\mathcal{F}_2)} H^*(-)) & \\
& \parallel & \\
& H^*(\mathcal{F}_1) \otimes H^*(\mathcal{F}_2) &
\end{array}$$

Since there are no finiteness issues we obtain via the universal property of inverse limits that $H^*(\mathcal{F}_1 \times \mathcal{F}_2) \cong H^*(\mathcal{F}_1 \otimes \mathcal{F}_2)$ in the category of unstable algebras over the Steenrod algebra. \square

The Kuenneth formula is natural in the following way.

Theorem 5.2 *Let $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ and $(S_2, \mathcal{F}_2, \mathcal{L}_2)$ be two p -local finite groups respectively. Then $H^*(\mathcal{F}_1 \times \mathcal{F}_2) \cong H^*(|\mathcal{L}_1 \times \mathcal{L}_2|_p^\wedge; \mathbb{F}_p) \cong H^*(|\mathcal{L}_1|_p^\wedge; \mathbb{F}_p) \otimes H^*(|\mathcal{L}_2|_p^\wedge; \mathbb{F}_p) \cong H^*(\mathcal{F}_1 \otimes \mathcal{F}_2)$.*

6 Glaubermann's and Thompson's theorems for p -local finite groups

In [7] Diaz, Glesser, Mazza and Park prove analogues of Glaubermann's and Thompson's theorems for fusion systems. We extend their results to p -local finite groups and give an algebraic criterion for the classifying space of a p -local finite group to be equivalent to BS before completion.

Theorem 6.1 *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. Then*

$$(Z(S))^p \cap Z(N_{\mathcal{F}}(J(S))) \cap Z(N_{\mathcal{L}}(J(S))) \leq Z(\mathcal{F}).$$

If p is odd or \mathcal{F} is S_4 -free, then $Z(\mathcal{F}) = Z(N_{\mathcal{F}}(J(S))) = Z(N_{\mathcal{L}}(J(S)))$.

Proof: In [7] the authors show that for $(S, \mathcal{F}, \mathcal{L})$ a p -local finite group we have $(Z(S))^p \cap Z(N_{\mathcal{F}}(J(S))) \leq Z(\mathcal{F})$ and if p is odd or \mathcal{F} is S_4 -free, then $Z(\mathcal{F}) = Z(N_{\mathcal{F}}(J(S)))$. The statement follows from the fact that for every p -local finite group $(S, \mathcal{F}, \mathcal{L})$ we have an equality $Z(\mathcal{F}) = Z(\mathcal{L})$. \square

Theorem 6.2 *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. Assume that p is odd or that \mathcal{F} is S_4 -free. If $C_{\mathcal{F}}(Z(S)) = N_{\mathcal{F}}(J(S)) = \mathcal{F}_S(S)$, and $N_{\mathcal{L}}(J(S)) = \mathcal{L}_S^c(S)$ then $\mathcal{F} = \mathcal{F}_S(S)$ and $|\mathcal{L}| \simeq BS$.*

Proof:

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